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Cycle factors in dense graphs

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Abstract

It is proved that a graph with n vertices and minimum degree at least $[(h+2)/2h]n$ contains $n/h - \mathrm{O}(h^2)$ vertex disjoint cycles of size h, and that a graph with $n > N(h)$ vertices and minimum degree at least $[(h+3)/2h]n$ contains n/h vertex disjoint h-cycles, provided h divides n. Bounds on the minimum degree required for G to contain a factor consisting of cycles of specified lengths are also discussed. This work is motivated by a conjecture of El-Zahar (1984).

1. Introduction

All graphs considered here are finite, undirected, and have neither loops nor parallel edges. The notation here follows the convention of [7] except when stated otherwise.

Let G denote a graph with n vertices, and let C_h denote the cycle of size h. Assuming that $n_1, \ldots, n_l$ satisfy $n_i \geqslant 3$ and $\sum_{i=1}^{l} n_i = n$, the question arises as to what minimum degree of G ensures that it contains vertex disjoint copies of $C_{n_1}, \ldots, C_{n_l}$.

It was conjectured by Sauer and Spencer [16] and proved by Aigner and Brandt [1] (see also [2]) that a minimum degree at least $\frac{2}{3}n$ suffices. The case of all n_i being equal to 3 was proved earlier by Corrádi and Hajnal [8].

A conjecture of El-Zahar [9] asserts that if among $n_1, \ldots, n_l$ there are exactly k odd numbers, a minimum degree at least $(n+k)/2$ suffices (this number is always an integer). This, if true, is tight, as shown for $k > 1$ by taking the graph G to be the complete 3-partite graph with two color classes A and B of size $(n-k)/2+1$ each, and one color class C of size $k-2$; this graph has n vertices, minimum degree $(n+k)/2-1$, and cannot contain k vertex disjoint odd cycles because each such cycle

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must contain a vertex from C. For $k \leqslant 1$, the lower bound is shown by taking G as the complete bipartite graph with one color class A of size $\lceil (n+1)/2 \rceil$ and one color class B of size $\lfloor (n-1)/2 \rfloor$, as every cycle in this graph occupies an equal number of vertices from A and B.

The conjecture was proved in its exact form only for several special cases. The case of all n_i being equal to 3 is the aforementioned theorem of [8], and the case $l=2$ was proved in [9]. The case where all n_i but n_1 are equal to 3 was proved more recently by Wang [18].

Some additional bounds are also known in more general cases. For example, it can be deduced from a result of Alon and Yuster [6] that if $n_1, \dots, n_l$ are all even, then a minimum degree $(\frac{1}{2} + \varepsilon)n$ suffices as long as $n > N(\varepsilon)$. A sketch of the proof is found in [2]. This means that for the case of all n_i being even, the El-Zahar conjecture is correct in an asymptotic sense. It was also proved in [2] that a minimum degree $(\frac{1}{2} + \varepsilon)n$ suffices if all n_i are larger than $N(\varepsilon)$ (and some of them may be odd).

The El-Zahar conjecture implies, in particular, that if $n_1, \dots, n_l$ are all equal to an odd constant number h, a minimum degree $[(h+1)/2h]n$ suffices for finding the required cycles in G. Here we prove that a slightly bigger minimum degree suffices.

It is assumed through most of the paper that h is an odd integer. In Section 2 we give an upper bound on the minimum degree required to find vertex disjoint copies of C_h which cover together most of the graph G, as stated in the following theorem.

Theorem 1.1. *If G is a graph with n vertices and minimum degree at least $[(h+2)/2h]n$, then G contains at least $n/h - (9h-4)h$ vertex disjoint C_h-copies.*

The error term in h appearing in this formulation can be reduced somewhat further, but not in a way which justifies calculating the exact value that the given proof can provide. The proof of this theorem is based on a variant of the proof method employed in [10] with some additional arguments. As the case in [10], the proof yields an efficient algorithm for finding these C_h-copies. Its presentation here is self-contained.

The following theorem shows for (odd) $h \geqslant 9$, that for n which is larger than a fixed function of h, and is divisible by h, a minimum degree at least $[(h+3)/2h]n$ ensures the existence of vertex disjoint C_h-copies which cover the whole graph G (for $h \leqslant 7$, a minimum degree at least $(2n-1)/3 < [(h+3)/2h]n$ guarantees, in particular, the existence of n/h vertex disjoint C_h-copies by [1]). The proof of this theorem uses the previous one as a lemma, and continues from there combining arguments similar to those used in the proofs in [5,6,2], with some additional arguments.

Theorem 1.2. *For every odd $h \geqslant 5$ there exists $N = N(h)$ such that if G is a graph with $n = kh > N$ vertices and minimum degree at least $[h/(2h-4)]n$, then G contains k vertex disjoint C_h-copies.*

This is proved in Section 4 (the coefficients here can be reduced somewhat further). No attempt is made to minimize $N(h)$. The proof uses certain lemmas connected with the Regularity lemma of Szemerédi, which are quoted in Section 3 along with some additional definitions and simple lemmas which are needed for the proof. In the course of the proofs we omit all floor and ceiling signs whenever the implicit assumption that a given quantity is an integer makes no essential difference.

Finally, Section 5 presents the following variant of Theorem 1.2, which deals with cases where n is large, and the required cycles are not necessarily of the same length, but instead are restricted to a fixed finite set of possible lengths. The possibility of some of these lengths being even is treated as well.

Theorem 1.3. *For every $\alpha>0$ and h there exists $N=N(\alpha,h)$ satisfying the following. Let G be a graph with $n>N$ vertices and minimum degree at least $[(1+14\alpha)/2]n$, and let $k_3,\dots,k_h$ be a list of nonnegative integers satisfying $\sum_{i=3}^{h} ik_i=n$ and $\sum_{j=1}^{(h-1)/2} k_{2j+1}\leqslant\alpha n$. Then G contains k_i copies of C_i for each i, all vertex disjoint.*

There is an additional cost in the required minimum degree of G, but it is still of the form $[(1+c\alpha)/2]n$, where αn is the number of odd cycles to be found in G, and c is a global constant (there is no attempt to minimize c in the proof). The proof of Theorem 1.3 uses a respective variant of Theorem 1.1, which is proved in the same section.

The paper ends with Section 6, containing some concluding remarks.

2. Almost cycle factors in sufficiently dense graphs

Proof of Theorem 1.1. We first prove the following proposition.

Proposition 2.1. *A graph G with n vertices and minimum degree at least $[(h+2)/2h]n+\frac{3}{2}h^2$ contains at least $n/h-5h$ vertex disjoint C_h copies.*

For Theorem 1.1 to follow, we add to our graph a clique (i.e. a complete graph) on $9h^2$ new vertices, connecting them to all others, use the proposition, and then remove the added clique along with all cycles containing any of its vertices.

Proof of Proposition 2.1. The following is a description of an algorithm which terminates only after the required C_h copies have been obtained. At each step we keep along with G a list $\mathscr{C}$ of vertex disjoint C_h copies which are contained in G, and a list $\mathscr{P}$ of vertex disjoint paths in G, some of which may be single vertices. At all stages, $\mathscr{C}$ and $\mathscr{P}$ will be vertex disjoint and will cover together all the vertices of G. In the beginning of the algorithm, $\mathscr{C}$ is empty and $\mathscr{P}$ holds all vertices of G, considered as paths (of length zero).

To each cycle from $\mathscr{C}$ we associate a weight of $2h+5$, and to each edge which is contained in a path from $\mathscr{P}$ we associate a weight of 2. The sum of all these weights, denoted by w, is never greater than $[(2h+5)/h]n$ (its initial value is zero).

We now describe an algorithm which at each stage increases the total weight w, and so must terminate after no more than $[(2h+5)/h]n$ stages. Moreover, at all times there will be no paths in $\mathscr{P}$ with more than $3h-7$ edges, and the algorithm will not terminate as long as there are more than $h-1$ paths with at least $h-2$ edges, or more than $2h-1$ paths of smaller lengths. In particular, $\mathscr{C}$ contains in the end at least $n/h-5h$ cycles.

At each stage of the algorithm we perform one of the following operations, depending on the current contents of $\mathscr{C}$ and $\mathscr{P}$.

- If there are at least h paths of length at least $h-2$ (i.e. with at least $h-2$ edges), we choose arbitrarily an $(h-2)$-subpath starting with an endvertex of each of these, and try to complete each such designated subpath to a cycle. If we can complete any one of these with a vertex of a member of $\mathscr{P}$ we do it, destroying up to $(h-1)+2=h+1$ edges ($h-1$ for the subpath itself if it is a proper subpath, another two for pulling a vertex out of another path, possibly splitting it in two), but getting a new cycle, thereby increasing w, by at least $(2h+5)-2(h+1)=3$.
- If there are h paths of length at least $h-2$, but their designated subpaths cannot be completed to cycles within the vertices of (the members of) $\mathscr{P}$, the counting argument given below shows that there is a cycle containing two vertices which close two of these subpaths to cycles (in fact, there is a cycle with three such vertices).

  Every designated subpath has at least $(2/h)n+3h^2$ vertices which are adjacent to its two endvertices (by virtue of the minimum degree of G), and no more than $h-3$ of these are found within $\mathscr{P}$ (only the vertices which are internal to the subpath itself can be thus connected in this setting). This means that there is a cycle which provides more than $2h$ ways of closing any one of the h subpaths, and so this cycle contains at least three distinct vertices which close three distinct subpaths.

  Destroying this cycle, closing two subpaths to new cycles, and keeping the (short) paths which are left from the original cycle, we increase the total number of cycles in $\mathscr{C}$ by one, and decrease the total number of edges in members of $\mathscr{P}$ by no more than $2(h-1)-(h-4)=h+2$. Thus w increases by at least 1.
- If there are no h long paths (i.e. paths of length at least $h-2$) in $\mathscr{P}$, and there are two endvertices of two short paths that can be connected, we connect them, adding an edge. Otherwise, if there are still at least $2h$ short paths, we partition these to pairs and for each pair we choose arbitrarily an endvertex of each path. We then try to find two neighboring vertices u, v on a path from $\mathscr{P}$ or a cycle from $\mathscr{C}$ such that u is adjacent to one of the designated vertices of our pair and v is adjacent to the other. Such a pair (u,v) of vertices is called a *connection* to our pair of paths. If any such connection is found within the members of $\mathscr{P}$ but outside the few paths longer than $h-3$ or the two paths of the pair to be connected itself, we connect the two

vertices to our two paths and gain an edge (we destroy only the edge between the two vertices that we found). This will leave fewer but longer paths. Note, however, that no path with more than $3h-7$ edges is created.

- If there are at least h pairs of short paths in $\mathscr{P}$, but they cannot be connected as described above within $\mathscr{P}$, there is a cycle with three distinct connections to three distinct pairs (the connections on the part of the cycle can be distinct without being disjoint). Here is a proof of this claim.

  Define a permutation α on the set of the vertices of G as follows. Attach to each cycle in $\mathscr{C}$ an arbitrary orientation, and let C' be an oriented cycle (not contained in G), which is disjoint from all the members of $\mathscr{C}$, but contains all the members of $\mathscr{P}$. α is defined by $\alpha(u)=v$ where v is the next vertex on the cycle containing u, which can be either C' or a member of $\mathscr{C}$. Denoting by v_1, v_2 the two designated vertices of one pair of paths, by virtue of the minimum degree of G there are at least $(2/h)n+3h^2$ vertices u such that u is adjacent to v_1 and $\alpha(u)$ is adjacent to v_2. In the setting here less than $3h^2$ of these are vertices of C'. This means that there is a member C of $\mathscr{C}$ which contains three distinct vertices which are attached to three distinct pairs in this manner, constituting together the three connections of our claim.

  We replace the edge between each of the three pairs of vertices thus found in C with the two edges of the appropriate connection. This replaces the cycle C and the three pairs of paths with three (longer) paths. Losing the cycle but gaining the new $h+3$ edges in $\mathscr{P}$ increases w again, by $2(h+3)-(2h+5)=1$. Again, no path with more than $3h-7$ edges is created.

This covers all possibilities, showing that the algorithm cannot terminate before the required number of vertex disjoint cycles is obtained. Thus, the proof of the proposition, and hence of Theorem 1.1 as well, is complete. □

3. Regularity and multicycles

For a graph G and disjoint sets A, B of vertices of G, the *density* of (A,B) is the number of edges from A to B divided by $|A||B|$. The pair (A,B) is called *γ-regular* if for any $A'\subset A$ and $B'\subset B$ satisfying $|A'|\geqslant\gamma|A|$ and $|B'|\geqslant\gamma|B|$, the densities of (A,B) and (A',B') differ by less than γ.

The relevance of the notion of regularity to existence proofs lies in the fact that a regular pair exhibits many of the properties of a random bipartite graph. As an example which is also used in what follows, let us quote the following lemma from [5], which enables us to deduce the existence of vertex disjoint copies of the complete h-partite graph with a vertices in each class from the regularity property.

Lemma 3.1 (Alon and Yuster [5]). *For any integers a, h, and any positive numbers δ, ε, there exist $N=N_{3.1}(h,a,\delta,\varepsilon)$ and $\gamma=\gamma_{3.1}(h,a,\delta,\varepsilon)>0$ such that every family of pairwise disjoint vertex sets $S_1,\ldots,S_h$ with $n>N$ vertices each, in which every* *pair* (S_i, S_j) *is* γ*-regular and of density at least* δ*, contains a subgraph with at least* $(1-\varepsilon)hn$ *vertices consisting of vertex disjoint copies of the complete h-partite graph with a vertices in each class.*

$N_{3.1}(h,a,\delta,\varepsilon)$ and $\gamma_{3.1}(h,a,\delta,\varepsilon)$ can be assumed to be monotone in a (otherwise replace $N_{3.1}(h,a,\delta,\varepsilon)$ with $\max\{N_{3.1}(h,b,\delta,\varepsilon) \mid 1 \leqslant b \leqslant a\}$, and $\gamma_{3.1}(h,a,\delta,\varepsilon)$ with the respective minimum). In what follows, all functions which can be assumed to be monotone in some of their arguments are assumed to be so.

The following lemma is a result of Szemerédi's Regularity lemma [17], and is proven implicitly in [5]. An explicit presentation of the proof is found in [3]. Similar statements have been proved and applied in various other existence proofs, such as [13] (see also the 'degree form' in [15]).

Lemma 3.2 (Alon and Yuster [5], see Alon and Fischer [3] and also Komlós and Simonovits [15]). *For any* $\varepsilon > 0$ *there exists* $\delta = \delta_{3.2}(\varepsilon)$*, such that for any k and* $\gamma > 0$ *there exists* $N = N_{3.2}(\varepsilon, k, \gamma)$ *satisfying all* α:

If G is a graph with $n > N$ *vertices and minimum degree at least* αn*, then there exists a partition* $C_0, \ldots, C_l$ *of G, with* $k \leqslant l \leqslant N$*, and a graph H with a vertex set* $\{v_1, \ldots, v_l\}$ *and minimum degree at least* $(\alpha - \varepsilon)l$*, such that* $|C_0| < \varepsilon n$*,* $C_1, \ldots, C_l$ *are of equal sizes, and if* v_i, v_j *are neighbors in H then* (C_i, C_j) *is a* γ*-regular pair in G with density at least* δ*.*

The strength of this lemma is in that it allows us to generalize an existence theorem through use of an appropriate lemma on the graph H, the *partition graph*, which it obtains. In our case, Theorem 1.1 is used as a lemma to find in G copies of the graph defined below.

A (*complete*) *multicycle with h classes*, $C_{a_1,\ldots,a_h}$, is a graph K with a vertex set which is the disjoint union of $V_1, \ldots, V_h$, such that $|V_i| = a_i$, with the edge set of K being all pairs (v,w) such that $v \in V_i$ and $w \in V_{i+1}$ for some $1 \leqslant i \leqslant h$ (the indices are taken modulo h). $V_1, \ldots, V_h$ are called the *cycle classes* of K. The relevance of multicycles for our purpose lies in the fact that they can be covered by vertex disjoint cycles and paths in many ways. For example, a multicycle with $h-2$ cycle classes of size h each contains $h-2$ vertex disjoint C_h copies (e.g. the one with classes $\{a_1, \ldots, a_5\}$, $\{b_1, \ldots, b_5\}$ and $\{c_1, \ldots, c_5\}$ contains $a_1b_1a_2b_2c_1a_1$, $b_3c_2b_4c_3a_3b_3$ and $c_4a_4c_5a_5b_5c_4$).

The following lemma allows us to deduce the existence of multicycles from the regularity property defined above. Here it is deduced from Lemma 3.1; it could also be proven for example using the Blow-up lemma [14].

Lemma 3.3. *For any sequence* $a_1, \ldots, a_h$*, all bounded by a, and any positive* δ*,* ε*, there exist* $N = N_{3.3}(h,a,\delta,\varepsilon)$ *and* $\gamma = \gamma_{3.3}(h,a,\delta,\varepsilon) > 0$ *such that every family of pairwise disjoint vertex sets* $S_1, \ldots, S_h$ *with* $n > N$ *vertices each, in which every pair* (S_i, S_{i+1}) *(the indices are taken modulo h) is* γ*-regular and of density at least* δ*, contains a subgraph with at least* $(1-\varepsilon)hn$ *vertices consisting of vertex disjoint* $C_{a_1,\ldots,a_h}$ *copies.*

Proof. It is enough to find such a subgraph consisting of vertex disjoint copies of the multicycle K with $\sum_{i=1}^{h} a_i$ vertices in each class, because each copy of K can be partitioned into h copies of $C_{a_1,\ldots,a_h}$. Set $N = N_{3.1}(h, ha, \delta, \varepsilon)$ and $\gamma = \gamma_{3.1}(h, ha, \delta, \varepsilon)$. Note that $ha \geqslant \sum_{i=1}^{h} a_i$. Assuming that the sets $S_1, \ldots, S_h$ of vertices from G satisfy the requirements here, add to G all possible edges from S_i to S_j for every (i,j) for which $|i-j|>1$ (modulo h). For the resulting graph G' Lemma 3.1 is applied; for each copy of the complete h-partite graph thus obtained in G' there corresponds a copy of K in G. □

From Theorem 1.1 it can be deduced that a graph G with $n > N_0(\varepsilon)$ vertices and minimum degree at least $((h+2)/2h - (1/7h)\varepsilon)n$ contains vertex disjoint C_h copies occupying together at least $(1-\varepsilon)n$ vertices. To prove this, add to G a clique on $(6/7h)\varepsilon n$ new vertices connecting them to all others, employ the theorem on the resulting graph, and then remove the added clique along with the cycles containing any of its vertices; N_0 is set to $(7/\varepsilon)(9h-4)h^2$.

By using this on the partition graph obtained through Lemma 3.2, and using Lemma 3.3 for each family of vertex sets corresponding to a cycle thus found in the partition graph, we find in G vertex disjoint multicycles, as stated in the following (for a detailed exposition of a similar proof, refer to [5]).

Lemma 3.4. *For any sequence $a_1, \ldots, a_h$, bounded by a, and any positive ε, there exists $N = N_{3.4}(h, a, \varepsilon)$ such that any graph G with $n > N$ vertices and minimum degree at least $[(h+2)/2h]n$ contains a subgraph consisting of vertex disjoint $C_{a_1,\ldots,a_h}$ copies, occupying together at least $(1-\varepsilon)n$ vertices.*

Proof (*Sketch*). Set

$$N = \frac{14h}{14h-\varepsilon} N_{3.2}\left(\frac{1}{14h}\varepsilon, N_0\left(\frac{1}{2}\varepsilon\right), \gamma_{3.3}\left(h, a, \delta_{3.2}\left(\frac{1}{14h}\varepsilon\right), \frac{3}{7}\varepsilon\right)\right)$$

$$\cdot N_{3.3}\left(h, a, \delta_{3.2}\left(\frac{1}{14h}\varepsilon\right), \frac{3}{7}\varepsilon\right),$$

assuming $\varepsilon < 1$. If G is a graph with $n > N$ vertices and minimum degree at least $[(h+2)/2h]n$, use Lemma 3.2, and then use the result of Theorem 1.1 on the partition graph, thereby partitioning G into subgraphs, one of which with at most $(1/14h + \frac{1}{2})\varepsilon n$ vertices, and all others satisfying the requirements of Lemma 3.3 needed to find vertex disjoint multicycles occupying all their vertices but at most $\frac{3}{7}\varepsilon n$ of them. These yield the lemma. □

In the proof of Theorem 1.2 we shall apply the last lemma with $h' = h - 2$ instead of h. Most C_h copies will then be found by the following lemma, after some preprocessing done with the aid of additional arguments. The simple proof of this lemma is left to the reader.

Lemma 3.5. *If v is a vertex in G which is adjacent to two vertices in one cycle class L of a multicycle K with $h-2$ classes, where K is contained in G and does not contain v, then G contains a C_h copy with one vertex being v, two vertices being taken from L, and one vertex being chosen arbitrarily from each of the other cycle classes of K.*

In Section 5, we shall also take C_h copies which are completely contained in the multicycles found through the application of the Regularity lemma.

The following simple lemma about the possibility of partitioning a graph keeping high degrees, is useful in many covering problems, and plays a role in the theorems to be proven here as well.

Lemma 3.6 (see e.g. Alon and Fischer [2] or Alon and Yuster [6]). *For any $\alpha>\beta>0$ and $\eta>0$ there exists $N=N_{3.6}(\alpha,\beta,\eta)$ satisfying the following. If G is a graph with $n>N$ vertices and minimum degree at least αn, and $n=k+l$ for k,l which are not smaller than ηn, then the vertex set of G can be partitioned into two sets A,B of sizes k,l, respectively, such that all vertices of G have at least βk neighbors in A and at least βl neighbors in B.*

Finally, we need the following lemma of Bondy.

Lemma 3.7 (see e.g. Bollobás [7]). *Any graph with n vertices and minimum degree exceeding $n/2$ contains a cycle of length h for every $3\leqslant h\leqslant n$.*

4. Cycle factors in sufficiently dense graphs

We begin with the following simple corollary of Lemma 3.4.

Corollary 4.1. *For any $a_1,\ldots,a_{h'}$, bounded by a, and any positive ε, there exists $N=N_{4.1}(h',a,\varepsilon)$ such that any graph G with $n>N$ vertices and minimum degree at least $((h'+2)/2h'-\varepsilon)n$ contains a subgraph, consisting of vertex disjoint $C_{a_1,\ldots,a_{h'}}$ copies, occupying a total of at least $(1-7ah'\varepsilon)n$ vertices.*

Proof. Set $N=N_{3.4}(h',a,ah'\varepsilon/(1+6\varepsilon))$. Add to G a clique on $6\varepsilon n$ new vertices, connecting them to all vertices of G. Use Lemma 3.4 on the resulting graph, and then remove the added clique along with any multicycles containing any of its vertices. □

Proof of Theorem 1.2. We assume that $h\geqslant 5$, and define $h'=h-2$, $a=h-1$. Assume that G is a graph with $n>N$ vertices, setting

$$N=\max\left\{\frac{(a+1)h'}{\varepsilon},N_{3.6}\left(\frac{h'+2}{2h'}-\varepsilon,\frac{h'+2}{2h'}-2\varepsilon,\frac{1}{2a+1}\right),2N_{4.1}(h',a,2\varepsilon)\right\}$$
$$+(a+1)h'$$

using

$$\varepsilon = \min\left\{\frac{1}{28a(a+1)h'}, \varepsilon_0\right\}$$

with ε_0 to be chosen later. Assume further that G has minimum degree at least $[(h'+2)/2h']n = [h/(2h-4)]n$, and that h divides n.

If $(a+1)h'$ does not divide n, let $0<k<h'=(a+1)h'/h$ be an integer such that $n'=n-kh$ is divisible by $(a+1)h'$, and use Lemma 3.7 repeatedly to find k vertex disjoint C_h copies one by one. Let G' be the graph obtained from G after these C_h copies are removed.

Use Lemma 3.6 to partition the vertex set of G' into two sets A and B, such that every vertex of G' has at least $((h'+2)/2h'-2\varepsilon)|A|$ neighbors in A and $((h'+2)/2h'-2\varepsilon)|B|$ neighbors in B, choosing

$$|A| = \frac{1}{(1-14ah'\varepsilon)}\frac{a}{(a+1)}n'$$

and appropriately

$$|B| = \frac{1-14a(a+1)h'\varepsilon}{(1-14ah'\varepsilon)(a+1)}n'.$$

Now apply Corollary 4.1 to A to find in it vertex disjoint copies of the multicycle with h' classes of size a each, occupying together $(1-14ah'\varepsilon)|A|$ of A's vertices. Let A' be the set of vertices contained in the multicycles thus obtained in A, and let B' be the union of B with the set of vertices of A not contained in any of these multicycles. Note that the number of the multicycles found on A' is equal to $(1/h')|B'|$.

Each vertex of G has at least

$$\left(\frac{h'+2}{2h'} - 2\varepsilon - 14ah'\varepsilon\right)|A| \geqslant \left(\frac{h'+2}{2h'} - 15ah'\varepsilon\right)|A'|$$

neighbors in A' and at least

$$\begin{aligned}\left(\frac{h'+2}{2h'} - 2\varepsilon\right)|B| &= \frac{|B|}{|B|+14ah'\varepsilon|A|}\left(\frac{h'+2}{2h'} - 2\varepsilon\right)|B'| \\ &> \frac{1}{1+28a^2h'\varepsilon}\left(\frac{h'+2}{2h'} - 2\varepsilon\right)|B'|\end{aligned}$$

neighbors in B'. A proper choice of ε_0 ensures that the coefficients of $|A'|$ and $|B'|$ in these expressions are both larger than $(h'+2)/(2h'+2) = (a+1)/2a$.

Now define a bipartite graph H whose color classes are the set of all cycle classes of the multicycles in A', and the set of vertices in B'. Define a vertex corresponding to a class L of a multicycle K in A' to be adjacent to a vertex v in B' if v is adjacent in G to at least two vertices from L. Observe that both color classes of H have the same number of members, $|B'|$.

Each vertex v in B' has at least $[(a+1)/2a]|A'|$ neighbors in G among the vertices of A'. Each cycle class which is not adjacent in H to v contains at most one vertex of

G which is a neighbor of v; each cycle class L which is adjacent in H to v contains at most $a=|L|$ neighbors of v. Thus, at least $\frac{1}{a-1}((a+1)/2a-1/a)|A'|=\frac{1}{2}|B'|$ of the classes are adjacent in H to v. A similar counting argument shows that each cycle class L of a multicycle in A' is adjacent in H to at least $\frac{1}{2}|B'|$ vertices, and so by Hall's Theorem (see e.g. [7]) H contains a perfect matching.

For each edge in this matching, which relates a class L of a multicycle K in A' to a vertex v in B', we use Lemma 3.5 to obtain a C_h copy. It contains v, two vertices from L, and one vertex chosen from each of the other cycle classes of K in a way that this procedure will remain applicable for the other edges of the matching as well. These C_h copies cover together G', completing the proof of the theorem. □

5. Specified cycles in sufficiently dense graphs

The following proposition, which is a crucial component of the proof of Theorem 1.3, is in some sense a generalization of Theorem 1.1. However, there is some additional cost in the minimum degree required of G.

Proposition 5.1. *Let C_i denote a cycle of size i, and let G be a graph with n vertices and minimum degree at least $\frac{1}{2}(n+3\sum_{i=3}^{h}k_i)+\frac{7}{2}h$, where $k_3,\dots,k_h$ satisfy $\sum_{i=3}^{h}ik_i\leqslant n-5h^2$. Then G contains a family of k_i copies of C_i for each i and a matching, all vertex disjoint, which cover together at least $n-3h$ of the vertices of G.*

Proof. The proof is similar to the proof of Proposition 2.1. For each i we keep a list $\mathscr{C}_i$ of vertex disjoint cycles of size i found in G, and we keep a list $\mathscr{P}$ of paths. At all times these lists are vertex disjoint and cover together G. This time, however, $\mathscr{C}_i$ is restricted to contain at each stage no more than k_i+2 cycles, while the paths in $\mathscr{P}$ are not restricted to any length. Note that in particular the members of $\mathscr{P}$ occupy together at least $3h^2$ vertices at all times.

To each edge contained in a path from $\mathscr{P}$ we attach a weight of 2 as before, and to each cycle from $\mathscr{C}_i$ we attach a weight of $2i+7$. As long as there are less than k_i cycles in any $\mathscr{C}_i$, or $\mathscr{P}$ contains at least $2h$ paths, we perform one of the following operations, increasing the total weight.

- If $\mathscr{C}_i$ contains less than k_i cycles, and there is any subpath of length $i-2$ of a path from $\mathscr{P}$ which can be closed within the vertices of $\mathscr{P}$ to a cycle of size i, we add the new cycle to $\mathscr{C}_i$ and keep in $\mathscr{P}$ the remaining paths. This time the subpath to be closed might not have a common endvertex with the path containing it, so the number of edges in the members of $\mathscr{P}$ may decrease by up to $i+2$ in this operation. The total weight, however, increases by at least $(2i+7)-2(i+2)=3$.
- Suppose now that $\mathscr{C}_i$ has less than k_i members, and there are h mutually vertex disjoint subpaths of length $i-2$ contained in the members of $\mathscr{P}$, such that none of them can be closed within $\mathscr{P}$ to a cycle (there is the possibility that several of them

are subpaths of the same long path from $\mathscr{P}$). Since every one of them can be closed to a cycle by at least $3\sum_{j=3}^{h} k_j + 6h > 3\sum_{j=3}^{h}(k_j + 2)$ vertices from G, a counting argument shows that there exists a cycle in $\mathscr{C}_j$ for some j, which provides more than $3h$ ways of closing any one of these subpaths to a cycle of size i. Hence, this cycle contains 4 distinct vertices closing 4 distinct subpaths to cycles (in particular $j > 3$).

Closing three of these subpaths to cycles, destroying for the purpose the aforementioned cycle from $\mathscr{C}_j$, but keeping the remaining paths, the total weight increases by at least $3(2i+7) - 6i - (2j+7) + 2(j-6) = 2$.

- If there are two endvertices of two paths in $\mathscr{P}$ which can be connected, we connect these paths, gaining an edge. Otherwise, assuming that $\mathscr{P}$ holds several paths, we choose an arbitrary orientation of each of the cycles of each $\mathscr{C}_i$ and an arbitrary direction of each of the paths of $\mathscr{P}$, and define a permutation α on the set of vertices of G as follows.

  Let C' be an oriented cycle (not contained in G) which contains exactly all the (now directed) paths of $\mathscr{P}$. For every vertex v of G, $\alpha(v)$ is the next vertex (according to the orientation) of the cycle that contains v, which is either C' or one of the cycles of $\mathscr{C}_j$ for some j.

  If w is a final vertex in one of the paths of $\mathscr{P}$, then in particular $\alpha(w)$ is an initial vertex in another path of $\mathscr{P}$, and there is no edge in G between w and $\alpha(w)$. A *connection* for such a w is a vertex v such that G contains an edge between v and w, and an edge between $\alpha(v)$ and $\alpha(w)$. If any such connection is found within $\mathscr{P}$ itself, by replacing the edge between v and $\alpha(v)$ with the two edges provided by the connection we increase the total number of edges in $\mathscr{P}$ and hence the total weight. Note that this way $\mathscr{P}$ remains a list of (vertex disjoint) paths.

  Suppose that no connection exists within $\mathscr{P}$, and $\mathscr{P}$ contains at least $2h$ paths. Choose h final vertices, $w_1, \ldots, w_h$, in such a way that no w_i shares a path with $\alpha(w_l)$ for some l. Since each w_i has more than $3\sum_{j=3}^{h}(k_j+2)$ connections in all, this implies that for some j there exists a cycle in $\mathscr{C}_j$ with four distinct connections to four distinct final vertices (in particular $j > 3$). Destroying this cycle and making the four connections, we increase the total weight again.

To summarize, because of the increase at each stage of the total weight, which is bounded, the algorithm must eventually terminate. However, this means that, in particular, $\mathscr{P}$ contains less than $2h$ paths, and so contains for every i at least h vertex disjoint $(i-2)$-subpaths (due to its number of vertices).

This in turn implies that every $\mathscr{C}_i$ must contain at least k_i members, so we have all the required cycles. A matching, covering all remaining vertices but at most $3h$ is found as a subgraph of the paths remaining in $\mathscr{P}$, and the surplus cycles (if there are any) in $\mathscr{C}_3, \ldots, \mathscr{C}_h$. $\square$

In order to treat the even cycles in Theorem 1.3, we use complete bipartite graphs instead of multicycles. Note that a complete bipartite graph with bg vertices in each class contains b vertex disjoint C_{2g} copies. We also need the following simple lemma, whose proof is left to the reader.

Lemma 5.2. *If K is a complete bipartite graph with color classes L_1, L_2, satisfying $|L_1| \geqslant g$ and $|L_2| \geqslant g-1$, and v is a vertex adjacent to two vertices in L_1, there exists a C_{2g} copy containing v, g vertices from L_1, and $g-1$ vertices from L_2.*

Proof of Theorem 1.3. We may assume that h is odd and larger than 3, and that $\alpha < \frac{1}{14}$. Set $N = \max\{N_1, N_2, N_3, N_4\}$, with all four N_i to be chosen later. Let a be a number which is larger than $2/\alpha$ such that $a+1$ is a common multiple of $3, \dots, h$. We first find k_3 vertex disjoint triangles one by one using Lemma 3.7. For each odd i we then find $0 \leqslant k < (i-2)(a+1)$ disjoint cycles of size i, with k chosen such that $(i-2)(a+1)$ divides $k_i - k$. For each even i we do the same with $0 \leqslant k < 2(a+1)$ chosen such that $2(a+1)$ divides $k_i - k$.

Denote by G' the remaining graph with n' vertices, and by k_i' the number of cycles of size i remaining to be found in G'. Define $\alpha_i' = k_i'/n'$, and $\alpha' = \sum_{j=2}^{(h-1)/2} \alpha_{2j+1}'$. Since $\alpha < \frac{1}{3}$, the taking of triangles does not increase the ratio of the number of odd cycles required to the number of remaining vertices, so this ratio remains bounded by α. Besides triangles, only a constant number (depending on h and α but not on n) of cycles is taken, and so N_1 can be chosen such that $\alpha' < \frac{5}{4}\alpha$. Note also that $\sum_{i=3}^{h} i\alpha_i' = 1$. N_2 is chosen such that G' has minimum degree at least $[(1+7\alpha)/2]n'$ (the taking of triangles reduces the minimum degree by no more than $(6\alpha/2)n$, and besides triangles only a constant number of vertices is taken). Note finally that $n' \geqslant \frac{1}{2}n$.

Choosing

$$N_3 = 2N_{3.6}\left(\frac{(1+7\alpha)}{2}, \frac{(1+6\alpha)}{2}, 1 - \frac{1}{(1-\eta)}\frac{a}{(a+1)}\right)$$

with $0 < \eta < 1/(a+1)$ to be chosen later, we now use Lemma 3.6 to partition the vertices of G' into two sets A and B with

$$|A| = \frac{1}{(1-\eta)}\frac{a}{(a+1)}n'$$

and appropriately

$$|B| = \frac{1-\eta(a+1)}{(1-\eta)(a+1)}n'.$$

We choose

$$N_4 = 4N_{3.2}\left(\frac{1}{2}\alpha, p, \gamma\right) \cdot t$$

with p, γ and t to be chosen later. Lemma 3.2 is applied to the subgraph of G' induced on A, let $l \geqslant p$ denote the number of vertices of the resulting partition graph I. Its minimum degree is at least $[(1+5\alpha)/2]l > [(1+4\alpha')/2]l$. Proposition 5.1 is then applied to I, finding

$$\left\lceil \left(1-\frac{1}{2}\eta\right)\frac{2j+1}{2j-1}\alpha'_{2j-1}l \right\rceil$$

cycles of size $2j-1$ for each $1<j\leqslant(h-1)/2$, and a matching with at least

$$\sum_{j=2}^{(h-1)/2}\left\lceil \left(1-\frac{1}{2}\eta\right)\frac{2j}{2}\alpha'_{2j}l \right\rceil$$

edges. Since

$$\sum_{j=2}^{(h-1)/2}(2j-1)\left(1-\frac{1}{2}\eta\right)\frac{2j+1}{2j-1}\alpha'_{2j+1}+\sum_{j=2}^{(h-1)/2}2\left(1-\frac{1}{2}\eta\right)\frac{2j}{2}\alpha'_{2j}=1-\frac{1}{2}\eta,$$

a proper choice of p will ensure that this is possible despite the ceiling signs appearing above, as well as that I has the required minimum degree.

Now, we find on the vertices of A, multicycles with a vertices in each class according to the cycles found in I, and complete bipartite graphs with a vertices in each class according to the edges of the matching. For each odd i we reserve $[i/(i-2)]k'_i/(a+1)$ multicycles with $i-2$ classes, and for each even i we reserve $(i/2)k'_i/(a+1)$ bipartite graphs, discarding all others. Note that exactly $\eta|A|$ of the vertices of A will not belong to the reserved multicycles and complete bipartite graphs; in order to find enough of these graphs we choose

$$t=\max\left\{N_{3.1}\left(2,a,\delta_{3.2}\left(\frac{1}{2}\alpha\right),\frac{1}{2}\eta\right),\right.$$
$$\left.\max\left\{N_{3.3}\left(2j-1,a,\delta_{3.2}\left(\frac{1}{2}\alpha\right),\frac{1}{2}\eta\right)\,\middle|\,2\leqslant j\leqslant\frac{h-1}{2}\right\}\right\}$$

and

$$\gamma=\min\left\{\gamma_{3.1}\left(2,a,\delta_{3.2}\left(\frac{1}{2}\alpha\right),\frac{1}{2}\eta\right),\right.$$
$$\left.\min\left\{\gamma_{3.3}\left(2j-1,a,\delta_{3.2}\left(\frac{1}{2}\alpha\right),\frac{1}{2}\eta\right)\,\middle|\,2\leqslant j\leqslant\frac{h-1}{2}\right\}\right\}.$$

We now define a bipartite graph H with vertex classes $\tilde{A}$ and $\tilde{B}$, with $\tilde{A}$ consisting of all cycle classes of the multicycles and color classes of the complete bipartite graphs obtained in A, and $\tilde{B}$ consisting of all the vertices not participating in members of $\tilde{A}$. A vertex v in $\tilde{B}$ and a class L in $\tilde{A}$ are defined to be adjacent in H if v is adjacent in G to at least two vertices from L. $\tilde{A}$ and $\tilde{B}$ have the same size, and the minimum degree of H can be ensured (by $a>2/\alpha$ and a proper choice of η) to exceed $\frac{1}{2}|\tilde{A}|=\frac{1}{2}|\tilde{B}|$ (one ensures that each vertex of G in a class of $\tilde{A}$ has more than $[(a+1)/2a]|\tilde{A}|$ neighbors in $\tilde{B}$, and that each vertex of $\tilde{B}$ has in G more than $[(a+1)/2a]a|\tilde{A}|$ neighbors within members of $\tilde{A}$; the choice of η depends on α and a alone so this is not cyclic). Thus, a perfect matching in H can be found using Hall's theorem.

Finally, for each edge in this perfect matching, a cycle is found by either Lemma 3.5 or Lemma 5.2. The smaller multicycles and complete bipartite graphs that remain after this contain the rest of the cycles needed to complete the proof of the theorem. □

6. Concluding remarks

- Theorem 1.3 gives information in many cases where the number of odd cycles that are needed to be found in G is small but not negligible. However, some cases are still outside the scope of Theorems 1.2 and 1.3. For example, let $n=h^2l$, with h being odd and l being even, and set $n_2,\dots,n_l$ equal to h and $n_1=n-h(l-1)$. Using the theorems stated here or the previously known statements (e.g. a repetitive usage of Lemma 3.7) one would prove that a minimum degree $[(h+\mathrm{O}(1))/2h]n$ suffices for G to contain vertex disjoint copies of $C_{n_1},\dots,C_{n_l}$, while according to the El-Zahar Conjecture a minimum degree $(n+l)/2=[(h^2+1)/2h^2]n$ should already suffice.
  A recent advance suggests that by using the Blow-up lemma of [14] or its algorithmic version from [12] instead of Lemmas 3.1 and 3.3, the scope of Theorem 1.3 can be extended by relaxing the restriction on the sizes of the cycles involved (e.g. relatively large even cycles can be found on regular pairs corresponding to the matching found on the partition graph; large odd cycles and any extra large cycles require a more specific treatment).
  Unfortunately, the amount of work involved in exploring the most general statement this method can provide takes it out of the scope of this paper, and has to be deferred to the future. The ultimate goal is to remove all restrictions on the size of the cycles (up to αn of which are odd) in Theorem 1.3, requiring n only to be larger than $N=N(\alpha)$.
- It would be interesting to investigate the cases where the number of odd cycles to be found is close but not equal to $\frac{1}{3}n$. For example, to prove or approximate the El-Zahar conjecture in case where all cycles to be found are triangles and squares. It would, of course, be interesting to prove the full conjecture.
- The proof of Theorem 1.1 (and Proposition 5.1) gives an efficient algorithm for finding the required cycles. The proof of Theorems 1.2 and 1.3 gives in theory an efficient algorithm too, using the algorithmic version of Szemerédi's regularity lemma given in [4].
  The coefficients of the running time of the algorithm for Theorems 1.2 and 1.3, as well as the constant N involved, however, are huge. A recent result of Gowers [11] shows that the coefficients and the constants which result from the full version

of the Regularity Lemma, applied here and in other existence proofs, cannot be significantly reduced.

Note added in proof. The author has learned that recently Sarmad Abbasi has been able to obtain strong results concerning the El-Zahar conjecture using the Regularity Lemma and the Blow-up Lemma (Private communication). These are currently being written and will hopefully appear in the near future.

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